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Prime Divisors, Asymptotic R-Sequences and Unmixed Local Rings

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INTRODUCTION

A current trend in commutative algebra has been to characterize properties of prime divisors in the completion of a local ring in terms of the independence of various elements and sequences in the local ring itself. For instance, in [2] Bruns characterizes the minimal depth of prime divisors of zero in the completion R^* of a local ring (R, M) as the largest number of elements which are M^n-independent, for n sufficiently large. More recently, the notion of asymptotic sequence has been used to characterize the minimal depth of minimal prime divisors of zero in R^*. This was done by the author in [5] and L. J. Ratliff, Jr. in [19]. (Elements $x_1,..., x_d$ form an asymptotic sequence if $x_{i+1} \notin \bigcup \{P \mid P \in \bar{A}^*(x_1,..., x_i)\}$, where $\bar{A}^*(I) =$ Ass $R/\overline{I^n}$ large n, and $\overline{I^n}$ is the integral closure of I^n.) Using this result as a starting point, a full-blown theory of asymptotic grade has been developed in [19] and [7].

In this paper we seek a similar characterization of the minimal depth of all prime divisors of zero in R^*. In particular, we introduce the set

$$\tilde{A}^*(I) = \bigcap \{A^*(J) \mid I^nR \subseteq JR \subseteq \overline{I^nR}, \text{ some } n\}$$

and show that $P \in \tilde{A}^*(I)$ if and only if there exists $\mathscr{P} \in \operatorname{Spec} T$ such that $(T_{\mathscr{P}})^*$ has a depth one prime divisor of zero, and $\mathscr{P} \cap R = P$. Here $A^*(J) = \operatorname{Ass} R/J^n$, large n and $T = R[It]$ is the Rees ring of R with respect to I, t an indeterminate. We then define the notion of asymptotic R-sequence: Elements $x_1,..., x_d$ form an asymptotic R-sequence if and only if $x_{i+1} \notin \bigcup \{P \mid \in \tilde{A}^*(x_1,..., x_i)\}$, $1 \leqslant i \leqslant d-1$. We prove in Theorem 3.4 that the length of any maximal asymptotic R-sequence characterizes the minimal depth among prime divisors of zero in R^*. It follows that R is

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unmixed if and only if each system of parameters forms an asymptotic R-sequence. A key point in our proofs is the finite generation of the ring $S=\sum_n I^n : \langle M \rangle t^n$ over the Rees ring T, for various ideals I ($I^n : \langle M \rangle = \bigcup_n (I^n : M^k)$). We accomplish this by showing that S is contained in certain ideal transforms over T, which themselves are finite. This relates to a problem posed by P. Schenzel at the end of [21]. He asks if R is unmixed, for which ideals I does it hold that S is a finite T-module? In Theorem 4.1 we show that R is unmixed if and only if S is a finite T-module for all ideals I with analytic spread less than dim R.

1. Notation and Preliminaries

In this section we record some of the standard facts and terminology that will be used throughout this paper. For definitions and facts from Noetherian ring theory not listed below, consult Nagata's well-known text [10].

Throughout R will denote a Noetherian commutative ring. In case R is local with maximal ideal M, we shall write R^* for the completion of R in the M-adic topology. Recall the following definitions:

1.1. Let R be a local ring. R is said to be unmixed (respectively, quasi-unmixed) in case $\dim R^*/Q^* = \dim R$ for each prime divisor of zero $Q^* \subseteq R^*$ (respectively, minimal prime divisor of zero).

1.2. Given an ideal $I \subseteq R$, we have the following sets of "asymptotic prime divisors of I":

(a) $A^*(I) = \operatorname{Ass} R/I^n$, large n.

(b) $\bar{A}^*(I) = \operatorname{Ass} R/\overline{I^n}$, large n.

By [1] and [14], respectively, the sets (a) and (b) are well-defined finite sets of primes. Here, by $\bar{J}$ we mean the integral closure of the ideal J. Recall that $\bar{J}$ is the largest ideal K containing J such that $JK^n = K^{n+1}$ for some n. (In this case J reduces K.) See [11] for details.

1.3. For an ideal $I \subseteq R$, $\mathscr{T}(I)$, the ideal transform of I, is the set of elements x in the total quotient ring of R such that $I^n x \subseteq R$ for some n. In what follows, we shall rely heavily upon the following result, due to J. Nishimura (see [12] and [13]):

THEOREM. *For a regular ideal $I \subseteq R$, $\mathscr{T}(I)$ is a finite R-module if and only if for each prime ideal P containing I, $(R_P)^*$ does not have a depth one prime divisor of zero.*

A notion related to the ideal transform, is a sort of "relative transform" studied successfully by Schenzel in [21]: Given ideals $I, J \subseteq R$, set $I: \langle J \rangle = \bigcup_n (I: J^n) = \{x \in R \mid J^n x \subseteq I$ for some $n \geqslant 1\}$. Of course $I: \langle J \rangle = (I: J^k)$ for k sufficiently large, and one easily checks that J never consists of zero divisors modulo $I: \langle J \rangle$.

1.4. Given an ideal $I \subseteq R$, we shall write $T = R[It]$ for the Rees ring of R with respect to I, t an indeterminate. Both T and $T[t^{-1}]$—the extended Rees ring—appear frequently throughout the literature. Virtually any statement that can be made about T has a corresponding statement concerning $T[t^{-1}]$. In particular, since primes $\mathscr{P} \in \operatorname{Spec} T$ containing IT are in one-to-one correspondence with primes $\mathscr{P}' \in \operatorname{Spec} T[t^{-1}]$ containing $t^{-1}T[t^{-1}]$, it is straightforward to check that $\mathscr{P} \in \operatorname{Ass} T/I^nT$ if and only if $\mathscr{P}' \in \operatorname{Ass} T[t^{-1}]/t^{-n}T[t^{-1}]$ and that $(T_{\mathscr{P}})^*$ has a depth one (minimal) prime divisor of zero if and only if $(T[t^{-1}]_{\mathscr{P}'})^*$ has a depth one (minimal) prime divisor of zero. Our preference for using T (instead of $T[t^{-1}]$) is twofold: First, T is sometimes called the "blowing-up" ring associated to I and is frequently studied in geometry. Hence results about T lend themselves to geometric interpretation. Secondly, and more to the point, we will have occasion to consider transforms of ideals in the Rees ring—a task made easier in the absence of negative grading. Finally, we shall frequently make use of the following well-known fact about T: If $J_1 \supseteq J_2 \supseteq \cdots$ is a filtration of ideals such that $I^n \subseteq J_n$ and $J_n J_m \subseteq J_{n+m}$ for all n and m, then $S = \sum_n J_n t^n$ is a finite T-module if and only if there exists $k > 0$ such that $I^n J_k = J_{n+k}$ for all $n \geqslant 1$ (see [6]).

1.5. Elements $(x_1, ..., x_d)$ are said to generate an ideal of the principal class if height $(x_1, ..., x_d) R = d$. Writing $I = (x_1, ..., x_d) R$, we have that $x_1, ..., x_d$ are analytically independent in R_P for each prime $P \supseteq I$. It follows that PT is prime for each such P, and that $T/PT \cong R/P[X_1, ..., X_d]$, the polynomial ring in d-variables over R/P.

1.6. The following condition on prime ideals will be called upon so frequently that we single it out here for ease of reference: Given a prime $P \in \operatorname{Spec} R$, we say that P satisfies condition (#) if and only if $(R_P)^*$ has a depth one prime divisor of zero. The reader should consult [16] for the first fundamental results given about local rings whose maximal ideal satisfies (#).

2. $\tilde{A}^*(I)$

In this section we introduce the set of prime ideals $\tilde{A}^*(I)$. We show that for a regular ideal I contained in the Noetherian ring R, $P \in \tilde{A}^*(I)$ if and only if there exists $\mathscr{P} \in \operatorname{Spec} T$ satisfying (#) with $\mathscr{P} \cap R = P$. We are motivated by a theorem of Ratliff which states that $P \in \bar{A}^*(I)$ if and only if there exists $\mathscr{P} \in \bar{A}^*(u\mathscr{R})$ with $\mathscr{P} \cap R = P$ (see [14]). Here $\mathscr{R}$ denotes the extended Rees ring $T[u]$, $u = t^{-1}$. By ([4], Lemmas 1.4 and 1.5) it follows that $\mathscr{P} \in \bar{A}^*(u\mathscr{R})$ if and only if there exists a minimal prime divisor of zero $q^* \subseteq (\mathscr{R}_{\mathscr{P}})^*$ such that $\mathscr{P} \cdot (\mathscr{R}_{\mathscr{P}})^*/q^* \in \bar{A}^*(u \cdot (\mathscr{R}_{\mathscr{P}})^*/q^*)$. By ([18, Theorem 2.12]) this holds if and only if height $\mathscr{P} \cdot (\mathscr{R}_{\mathscr{P}})^*/q^* = 1$ (since $(\mathscr{R}_{\mathscr{P}})^*/q^*$ is quasi-unmixed). In light of these remarks and 1.4, we may rephrase Ratliff's result as follows:

> $P \in \bar{A}^*(I)$ if and only if there exists $\mathscr{P} \in \operatorname{Spec} T$ with $\mathscr{P} \supseteq IT$ such that $(T_{\mathscr{P}})^*$ has a depth one minimal prime divisor of zero and $\mathscr{P} \cap R = P$.

We now define $\tilde{A}^*(I)$:

Definition. Given an ideal $I \subseteq R$ and a prime $P \supseteq I$, $P \in \tilde{A}^*(I)$ if and only if $P \in \bigcap A^*(J)$, where the intersection ranges over all ideals J such that $I^n \subseteq J \subseteq \overline{I^n}$ for some n. It follows from standard localization arguments that $P \in \tilde{A}^*(I)$ if and only if $PR_P \in \tilde{A}^*(IR_p)$. It is easy to check that $\bar{A}^*(I) \subseteq \tilde{A}^*(I) \subseteq A^*(I)$, because in [14] it is shown that $\bar{A}^*(I) \subseteq A^*(I)$.

We will need the following lemmas to characterize $\tilde{A}^*(I)$.

2.1 Lemma. *Let $R \subseteq S$ be a finite integral extension. Assume that R is a domain and that S is R torsion-free. Then $P \in \operatorname{Spec} R$ satisfies (#) if and only if there exists $Q \in \operatorname{Spec} S$ satisfying (#) with $P = Q \cap R$.*

(*Note*: only the if part of the above statement requires that R be domain.)

Proof. Suppose $Q \in \operatorname{Spec} S$ satisfies (#) and $P = Q \cap R$. We may assume that R is local at P. It follows that S is semi-local with Jacobson radical $J = Q \cap Q_2 \cap \cdots \cap Q_r$, where $Q, Q_2, ..., Q_r$ are the prime ideals of S lying over P. Because $R \subseteq S$ is a finite integral extension, the completion S^* of S in the J-adic topology is $R^* \otimes S$, where R^* is the completion of R in the P-adic topology. On the other hand, since S is semi-local, $S^* \cong (S_Q)^* \oplus \cdots \oplus (S_{Q_r})^*$, so there exists a prime divisor of zero $q^* \subseteq S^*$ with height $Q^*/q^* = 1$ (by assumption on Q). It follows that $R^*/q^* \cap R^* \subseteq S^*/q^*$ is an integral extension over a complete local domain. Since $R^*/q^* \cap R^*$ is Henselian ([10]) it follows that S^*/q^* is local. That is, Q^* is the only maximal ideal in S^* containing q^*. It follows from this that dim $R^*/q^* \cap R^* = 1$. Since P is regular, P^* does not belong to Ass R^*. Since $q^* \cap R^*$ consists of zero divisors (by [10, 18.12]), and dim $R^*/q^* \cap R^* = 1$, we have $q^* \cap R^* \in \operatorname{Ass} R^*$, as desired.

Conversely, suppose that $R \subseteq S$ is finite and $P \in \operatorname{Spec} R$ satisfies (#). We may assume that R is local at P. By 1.3 we have that $\mathscr{T}(P)$, the ideal transform of P, is not a finite R-module. Letting J = Jacobson radical of S, it follows that $\mathscr{T}(PS) = \mathscr{T}(J)$. If $\mathscr{T}(J)$ were a finite S-module, then $\mathscr{T}(PS)$ would be a finite S-module, and hence a finite R-module. But $\mathscr{T}(P) \subseteq \mathscr{T}(PS)$, and this would be a contradiction. Therefore $\mathscr{T}(J)$ is an infinite S-module. Hence by 1.3 Q satisfies (#) for some $Q \in \operatorname{Spec} S$ containing J. Of course $Q \cap R = P$.

2.2 Lemma. *Let $P \in \operatorname{Spec} R$ and $I \subseteq R$ be a regular ideal. Suppose $I \subseteq P$, and $n > 0$. Set $T = R[It]$ and $T' = R[I^n t]$. If there exists $\mathscr{P} \in \operatorname{Spec} T$ such that $\mathscr{P}$ satisfies (#) and $\mathscr{P} \cap R = P$, then there exists $\mathscr{P}' \in \operatorname{Spec} T'$ satisfying (#) with $\mathscr{P}' \cap R = P$.*

Proof. Following Ratliff in [15], we set $A = R[I^n t^n]$. Then there exists an isomorphism of A with T' that fixes R. As in the proof of Theorem 2.5 below, we may assume that R is a domain. Since T is integral over A and $T = A[It]$, T is A-finite, so Lemma 2.1 applies.

2.3 Lemma. *Let $I \subseteq P \in \operatorname{Spec} R$. Set $T = R[It]$ and $T' = R_P[I_P t]$. Then there exists $\mathscr{P}' \in \operatorname{Spec} T'$ satisfying (#) with $\mathscr{P}' \cap R_P = PR_P$ if and only if there exists $\mathscr{P} \in \operatorname{Spec} T$ satisfying (#) with $\mathscr{P} \cap R = P$.*

Proof. Since localization commutes with formation of the Rees ring, the result follows.

2.4 Lemma. *Let $R \subseteq S$ be a faithfully flat extension of Noetherian rings. There exists $P \in \operatorname{Spec} R$ satisfying (#) if and only if there exists $Q \in \operatorname{Spec} S$ satisfying (#) with $Q \cap R = P$.*

Proof. Suppose $P \in \operatorname{Spec} R$ satisfies (#). We may localize at P, and by 1.3, assume that $\mathscr{T}(P)$ is an infinite R-module. Therefore $\mathscr{T}(P) \otimes_R S = \mathscr{T}(PS)$ is an infinite S-module. By 1.3 Q satisfies (#) for some $Q \in \operatorname{Spec} S$ containing PS. Of course $Q \cap R = P$. The converse is similar.

2.5 Theorem. *Let $I \subseteq R$ be a regular ideal and $P \in \operatorname{Spec} R$ with $I \subseteq P$. Then $P \in \tilde{A}^*(I)$ if and only if there exists $\mathscr{P} \in \operatorname{Spec} T$ satisfying (#) with $\mathscr{P} \cap R = P$.*

Proof. By Lemma 2.3 and the definition of $\tilde{A}^*(I)$, we may assume that R is local at P. Now suppose $P \in \tilde{A}^*(I)$. If $P \in \bar{A}^*(I)$, then Ratliff's result implies that there exists $\mathscr{P} \in \operatorname{Spec} T$ such that $(T_{\mathscr{P}})^*$ has a depth one minimal prime divisor of zero, and $\mathscr{P} \cap R = P$. Therefore we may assume that $P \notin \bar{A}^*(I)$. Consequently, for all large k, $I^k\colon \langle P \rangle \subseteq \overline{I^k}$, so $P \in A^*(I^k\colon \langle P \rangle)$. If $\mathscr{P}$ does not satisfy (#) for all $\mathscr{P} \in \operatorname{Spec} T$ containing PT, then 1.3 implies that $\mathscr{T}(PT)$, the ideal transform of PT, is a finite T-module. Consider the ring $S = \sum_n I^n\colon \langle P \rangle t^n$. The following facts imply $S \subseteq \mathscr{T}(PT)$:

(i) $(PT)^j = \sum_n P^j I^n t^n$ for all j.

(ii) If $I^n\colon \langle P \rangle = (I^n\colon P^k)$ then $(PT)^k \cdot (I^n\colon \langle P \rangle)\, t^n \subseteq T$.

Therefore S is a finite T-module. It follows that there exists $k' > 0$ such that $I^{n+k'}\colon \langle P \rangle = I^n(I^{k'}\colon \langle P \rangle)$ for all $n \geqslant 1$. By choosing k and k' large enough we may assume $k = k'$. Then easily $I^{nk}\colon \langle P \rangle = (I^k\colon \langle P \rangle)^n$ for all n, and since $P \in \operatorname{Ass} R/(I^k\colon \langle P \rangle)^n$ for all large n, we have a contradiction. It follows that $\mathscr{P}$ satisfies (#) for some $\mathscr{P} \supseteq PT$, and $\mathscr{P} \cap R = P$.

Conversely, suppose there exists $\mathscr{P} \in \operatorname{Spec} T$ satisfying (#) with $\mathscr{P} \cap R = P$. Let $n > 0$ and suppose $I^n \subseteq J \subseteq \overline{I^n}$. We need to show $P \in A^*(J)$. Now I^n reduces J, therefore $T' = R[Jt]$ is a finite $T'' = R[I^n t]$-module. By Lemma 2.2 there exists $\mathscr{P}'' \in \operatorname{Spec} T''$ satisfying (#) with $\mathscr{P}'' \cap R = P$. By Lemma 2.1 there exists $\mathscr{P}' \in \operatorname{Spec} T'$ satisfying (#) with $\mathscr{P}' \cap R = P$. Because $T' \subseteq T' \otimes R^*$ is a faithfully flat extension and $T' \otimes R^*$ is $R^*[JR^*t]$ and because $PR^* \in A^*(JR^*)$ if and only if $P \in A^*(J)$ (Lemma 2.4 and [10, 18.11]), we may assume that R is complete. Now ([16]) implies that $\mathscr{P}' \in A^*(JT')$. If we show that $\mathscr{P}'$ is relevant (i.e., $Jt \nsubseteq \mathscr{P}'$) then ([14, 2.6.1]) implies that $\mathscr{P}' \cap R = P \in A^*(J)$.

Now, because $\mathscr{P}'$ satisfies (#) there exists $q^* \in \operatorname{Ass}(T'_{\mathscr{P}'})^*$, a depth one prime divisor of zero. Let $q = q^* \cap T'$ and $Q = q^* \cap R$. Then $q \in \operatorname{Ass} T'$, $Q \in \operatorname{Ass} R$ and $T'/q =$ Rees ring of R/Q with respect to $JR + Q/Q$ (since $q = QR[t] \cap T'$). Moreover $(T'_{\mathscr{P}'})^*/q(T'_{\mathscr{P}'})^* \cong (T'_{\mathscr{P}'}/qT'_{\mathscr{P}'})^*$ and $q^*/q(T'_{\mathscr{P}'})^*$ is a depth one prime divisor of zero in this ring. If $\mathscr{P}'$ were irrelevant in T' then the image of $\mathscr{P}'$ in T'/q' would be irrelevant as well. Therefore we may assume further that R is a complete local domain. But then R is unmixed, so $T'_{\mathscr{P}'}$ is unmixed ([9, 4.7]). Since $\mathscr{P}'$ satisfies (#), this forces $\dim(T'_{\mathscr{P}'})^* = 1$, so height $\mathscr{P}' = 1$. Since the altitude formula holds between R and T', this implies that height $\mathscr{P}' =$ height $P + \operatorname{trdeg}_R T' - \operatorname{trdeg}_{R/P} T'/\mathscr{P}'$. Assuming that $\mathscr{P}'$ were irrelevant, this would imply $\operatorname{trdeg}_{R/P} T'/\mathscr{P}' = 0$. Hence we would have height $\mathscr{P}' = 1 =$ height $P + 1 - 0$, so height $P = 0$. Since I contains non-zero divisors, this is absurd. It follows that $\mathscr{P}'$ is relevant, so $P \in A^*(J)$ as desired.

2.6 Corollary. *Let $R \subseteq S$ be a faithfully flat extension of Noetherian rings, and suppose $I \subseteq R$ is a regular ideal. Then $P \in \tilde{A}^*(I)$ if and only if there exists $Q \in \tilde{A}^*(IS)$ with $Q \cap R = P$.*

Proof. Immediate from Lemma 2.4 and Theorem 2.5.

2.7 Corollary. *Let $I \subseteq J \subseteq R$ be regular ideals and suppose I reduces J. Then $\tilde{A}^*(I) = \tilde{A}^*(J)$.*

Proof. I reduces J if and only if $T' = R[Jt]$ is a finite $T = R[It]$-module. Therefore the result follows from Lemma 2.1 and Theorem 2.5.

2.8 Corollary. *If R is local and R^* has no embedded prime divisors of zero, then $\bar{A}^*(I) = \tilde{A}^*(I)$ for all ideals I with height $I > 0$.*

It is not difficult to show that if R is a local ring such that R^* has no embedded prime divisors of zero and $Q \in \operatorname{Spec} T$, where T is a Rees ring, then $(T_Q)^*$ has no embedded divisors of zero. The result now follows from Ratliff's result above and Theorem 2.5.

2.9 Corollary. *Let $I \subseteq R$ be a regular ideal. The asymptotic prime divisors of I fit the following scheme:*

(1) *$P \in A^*(I)$ if and only if there exists relevant $\mathscr{P} \in \operatorname{Ass} T/IT$ with $\mathscr{P} \cap R = P$.*

(2) *$P \in \tilde{A}^*(I)$ if and only if there exists relevant $\mathscr{P} \in \operatorname{Ass} T/IT$ such that $\mathscr{P}$ satisfies (#) and $\mathscr{P} \cap R = P$.*

(3) *$P \in \bar{A}^*(I)$ if and only if there exist relevant $\mathscr{P} \in \operatorname{Ass} T/IT$ such that $(T_{\mathscr{P}})^*$ has depth one minimal prime divisor and $\mathscr{P} \cap R = P$.*

Proof. (1) is given in [14], (2) follows from Theorem 2 and its proof (since $IT_{\mathscr{P}}$) is principal and (3) is explained above.

3. Asymptotic R-Sequences

In this section we define asymptotic R-sequences and show that the length of all maximal asymptotic R-sequences coincides with $\min\{\dim R^*/Q^* \mid Q^* \in \operatorname{Ass} R^*\}$, whenever R is local with completion R^*. In fact, we shall assume throughout this section that R is a local ring with maximal ideal M and completion R^*.

Recall that elements $x_1,..., x_d$ form an R-sequence if and only if for all $1 \leqslant i \leqslant d-1$, $x_{i+1} \notin \bigcup \{P \mid P \in \operatorname{Ass} R/(x_1,..., x_i) R\}$. If I is an ideal generated by an R-sequence, then it is well known that I^n/I^{n+1} is a free R/I-module for all n. It follows easily from this that $\operatorname{Ass} R/I^n = \operatorname{Ass} R/I$ for all n (see [3, Exercise 3-1, 13]). Therefore elements $x_1,..., x_d$ form an R-sequence if and only if for all $1 \leqslant i \leqslant d-1$, $x_{i+1} \notin \bigcup \{P \mid P \in A^*((x_1,..., x_d) R)\}$. This motivates the definition:

3.1 Definition. Elements $x_1,..., x_d$ are said to form an asymptotic R-

sequence (ARS) provided $x_{i+1} \notin \bigcup \{P \mid P \in \tilde{A}^*((x_1,..., x_i) R)\}$ for all $1 \leqslant i \leqslant d-1$. An ARS $x_1,..., x_d$ is said to be maximal if $M \in \tilde{A}^*((x_1,..., x_d) R)$. Note that when $d=1$ we are requiring that x_1 be a non-zero divisor.

3.2 PROPOSITION. *Let R be a complete local ring and $I \subseteq R$ an ideal. Supose that for all $Q \in \operatorname{Ass} R$, $IR + Q/Q$ is an ideal of the principal class of height d. If there exists $\mathscr{P} \in \operatorname{Spec} T$ satisfying $(\#)$ with $\mathscr{P} \supseteq IT$, then there exists $Q \in \operatorname{Ass} R$ such that $P = \mathscr{P} \cap R$ is minimal over $I + Q$.*

Proof. Suppose $\mathscr{P} \in \operatorname{Spec} T$ and $q^* \in \operatorname{Ass}(T_{\mathscr{P}})^*$ is a depth one prime divisor of zero. Then arguing as in the proof of Theorem 2.5, we let $q = q^* \cap T$ and $Q = q^* \cap R$ and we reduce to the case that R is a complete local domain, as before (by noting that T/q is the Rees ring of R/Q with respect to $IR + Q/Q$). Hence I is an ideal of the principal class of height d and we must show that $P = \mathscr{P} \cap R$ is minimal over I. But R is unmixed, so $T_{\mathscr{P}}$ is unmixed. Therefore if $\mathscr{P}$ satisfies $(\#)$, we have height $\mathscr{P} = 1$. But PT is a non-zero prime ideal contained in $\mathscr{P}$, so $\mathscr{P} = PT$ and $\operatorname{trdeg}_{R/P} T/\mathscr{P} = \operatorname{trdeg}_{R/P} T/PT = d$ (by 1.5). By the altitude formula ([6]):

height $\mathscr{P}$ = height $P + \operatorname{trdeg}_R T - \operatorname{trdeg}_{R/P} T/PT$, so height $P = d$ and P is minimal over I, as desired.

3.3 PROPOSITION. *Let $I \subseteq R$ be a regular ideal. Assume that for all $Q^* \in \operatorname{Ass} R^*$, $IR^* + Q^*/Q^*$ is an ideal of the principal class of height d. If $P \in \tilde{A}^*(I)$, then there exist primes $P^*, Q^* \subseteq R^*$ such that $Q^* \in \operatorname{Ass} R^*$, P^* is minimal over $IR^* + Q^*$ and $P^* \cap R = P$.*

Proof. By Theorem 2.5 there exists $\mathscr{P} \in \operatorname{Spec} T$ with $\mathscr{P} \supseteq IT$, $\mathscr{P}$ satisfies $(\#)$ and $\mathscr{P} \cap R = P$. By Lemma 2.4 there exists $\hat{\mathscr{P}} \in \operatorname{Spec} \hat{T}$ satisfying $(\#)$ with $\hat{\mathscr{P}} \cap T = \mathscr{P}$, where $\hat{T} = T \otimes_R R^*$, a faithfully flat extension of T. Noting that $\hat{T}$ is the Rees ring of R^* with respect to IR^*, we may apply Proposition 3.2 to finish the proof (by taking $P^* = \hat{\mathscr{P}} \cap R^*$).

3.4 THEOREM. *Elements $x_1,..., x_d$ in R form an* ARS *if and only if for all $Q^* \in \operatorname{Ass} R^*$ it holds that $(x_1,..., x_d) R^* + Q^*/Q^*$ is an ideal of the principal class of height d. It follows that any permutation of an ARS remains an* ARS. *Moreover, the lengths of all maximal* ARS's *are the same and may be computed as*

$$\min\{\dim R^*/Q^* \mid Q^* \in \operatorname{Ass} R^*\}.$$

Proof. If the condition holds for all $Q^* \in \operatorname{Ass} R^*$, then height $(x_1,..., x_i) R^* + Q^*/Q^* = i$ for all $1 \leqslant i \leqslant d$ and $Q^* \in \operatorname{Ass} R^*$, because complete local domains are catenary. It follows from Proposition 3.3 that $x_1,..., x_d$ form an ARS.

Conversely, suppose that for some $i+1 \leqslant d$, there exist primes P^*, $Q^* \subseteq R^*$ with $Q^* \in \mathrm{Ass}\, R^*$, P^* minimal over $(x_1,..., x_i) R^* + Q^*$ and $x_{i+1} \in P = P^* \cap R$. Let $I = (x_1,..., x_i)$. Then for any $n > 0$ and ideal J such that $I^n R \subseteq JR \subseteq \overline{I^n R}$, it holds that P^* is minimal over $JR^* + Q^*$. By ([7, Proposition 1.13]) it follows that $P^* \in A(JR^*)$. Faithful flatness implies that $P^* \cap R = P \in A^*(J)$ ([10, 18.11]). Hence $P \in \tilde{A}(I)$ and $x_1,..., x_{i+1}$ do not form an asymptotic R-sequence.

That any permutation of an ARS remains an ARS is now clear, as is the inequality $\min\{\dim R^*/Q^* \mid Q^* \in \mathrm{Ass}\, R^*\} \geqslant$ length of any maximal ARS. The reverse inequality follows from Proposition 3.3.

We now characterize unmixed local rings in terms of asymptotic R-sequences.

3.5 THEOREM. *Let R be a local ring. The following are equivalent.*

(i) *R is unmixed.*

(ii) *For each ideal of the principal class I and* $P \in \tilde{A}^*(I)$, *it holds that* height $P =$ height I.

(iii) *Every system of parameters forms an* ARS.

(iv) *Some system of parameters forms an* ARS.

Proof. (i) implies (ii). Suppose that R is unmixed. Let I be an ideal of the principal class of height d and $P \in \tilde{A}^*(I)$. Theorem 2.5 implies there exists $\mathscr{P} \in \mathrm{Spec}\, T$ satisfying (#) with $\mathscr{P} \cap R = P$. Because R is unmixed, $T_{\mathscr{P}}$ is unmixed, so height $\mathscr{P} = 1$. Hence $PT = \mathscr{P}$ and the altitude formula implies height $P = d$.

(ii) implies (iii). If (ii) holds then there certainly exists an ARS of length $= \dim R$, so R is unmixed by Theorem 3.4. Like catenary local domains, unmixed local rings have the property that if $I = (x_1,..., x_d) R$ is an ideal of the principal class, then $(x_1,..., x_i) R$ is an ideal of the principal class for each $1 \leqslant i \leqslant d$. (iii) now follows from (ii).

(iii) implies (iv). Obvious.

(iv) implies (i). Immediate from Theorem 3.4.

In [2] it is shown that if $x_1,..., x_d$ in R satisfy the condition given in Theorem 3.4, then for all $n \geqslant 1$ there exists an i such that $x_1^i,..., x_d^i$ are M^n-independent (i.e., any form $f(x)$ vanishing at $x_1^i,..., x_d^i$ must have coefficients in M^n). It follows from the above that if R is unmixed, that this holds for any $x_1,..., x_d$ in R generating an ideal of the principal class. We record this as a corollary.

3.6 COROLLARY. *Let R be an unmixed local ring and suppose $x_1,..., x_d$ generate an ideal of the principal class of height d in R. Then for all $n \geqslant 1$ there exists an i such that $x_1^i,..., x_d^i$ are M^n-independent.*

Proof. It follows from Theorems 3.4 and 3.5 that for all $Q^* \in \operatorname{Ass} R^*$ it holds that height $((x_1,..., x_d) R^* + Q^*/Q^*) = d$. Now Theorem 1, [2] finishes the proof.

4. A PROBLEM OF P. SCHENZEL

In [21] Schenzel was able to characterize unmixed local rings by comparing the topologies defined by I and I^n: $\langle M \rangle$, $n \geqslant 1$. In particular, he proved that a local ring (R, M) is unmixed if and only if for each ideal I of the principal class with height $I < \dim R$, I^n: $\langle M \rangle$, $n \geqslant 1$, defines the I-adic topology. Motivated by the classical paper [20] of Rees, he posed the following problem.

PROBLEM (cf. [21]). If (R, M) is an unmixed local ring for which ideals $I \subseteq R$ does it hold that $S = \sum_n I^n$: $\langle M \rangle$ t^n is a finite T-module?

We shall answer this question using our work from sections two and three.

4.1 THEOREM. *Let (R, M) be a local ring. The following are equivalent:*

(i) *R is unmixed.*

(ii) *For each ideal of the principal class I, with* height $I < \dim R$, *S is a finite T-module.*

(iii) *For each ideal I with analytic spread $I < \dim R$, S is a finite T-module.*

Proof. (i) implies (ii). Suppose that R is unmixed and $I \subseteq R$ is an ideal of the principal class with height $I = d < \dim R$. If there exists $\mathscr{P} \in \operatorname{Spec} T$ such that $\mathscr{P}$ satisfies (#) and $\mathscr{P} \cap R = M$, then because $T_{\mathscr{P}}$ is unmixed, height $\mathscr{P} = 1$, $MT = \mathscr{P}$ and the altitude formula implies height $M = d$, a contradiction. Therefore by 1.3 $\mathscr{T}(MT)$ is a finite T-module. Since $S \subseteq \mathscr{T}(MT)$, the result follows.

(ii) implies (i). Suppose that (ii) holds and R is not unmixed. Let $x_1,..., x_d$ be a maximal ARS and write $I = (x_1,..., x_d) R$. Then $M \in \tilde{A}^*(I)$ and $d < \dim R$ by Theorem 3.5. By assumption S is a finite T-module, so there exists $k > 0$ such that I^{n+k}: $\langle M \rangle = I^n(I^k$: $\langle M \rangle)$ for all $n \geqslant 1$. It follows that $\bigcap I^n$: $\langle M \rangle = 0$.

On the other hand, by Theorem 3.4 and Proposition 3.3 there exists

$Q^* \in \operatorname{Ass} R^*$ such that MR^* is minimal over $IR^* + Q^*$. Now suppose $x \in R^*$ is such that for all $n \geqslant 1$ there exists $j > 0$ (depending on n) with $xM^jR^* \subseteq I^nR^* + Q^*$. Then $xyM^jR^* \subseteq I^nR^*$, where $Q^* = (0 :_{R^*} y)$. Then $xy \in \bigcap_n I^nR^* : \langle MR^* \rangle = 0$, so $x \in Q^*$. Therefore we may assume that R is a local domain with an M-primary ideal I such that $\bigcap_n I^n : \langle M \rangle = 0$. Because $M^{jn} \subseteq I^n$ for some j, all n, $1 \in \bigcap_n I^n : \langle M \rangle = 0$, and this is a contradiction.

(i) implies (iii). Recall that the analytic spread of I equals $\operatorname{trdeg}_{R/M} T/MT$. Now it does no harm to assume that R/M is infinite. Hence, by the Northcott–Rees theory of reductions [11], there exists a minimal reduction J of I generated by d analytically independent elements (d = analytic spread of I). Let $T' = R[Jt]$ and $S' = \sum_n J^n : \langle M \rangle t^n$. Since $T'/MT' \cong R/M[X_1, ..., X_d]$, the polynomial ring in d variables over R/M, MT' is a prime ideal and the arguments employed in the proof of (i) implies (ii) show that S' is a finite T' module when $d < \dim R$.

Now, because J reduces I, there exists $k > 0$ such that $I^{n+k} = J^nI^k$ for all $n \geqslant 1$. Therefore $S = \sum_n I^n : \langle M \rangle t^n = \sum_{j=0}^k I^j : \langle M \rangle t^j + \sum_{n=1}^{\infty} J^nI^k : \langle M \rangle t^{n+k} \subseteq \sum_{j=0}^k I^j : \langle M \rangle t^j + S' \cdot t^k \subseteq \sum_{j=0}^k I^j : \langle M \rangle t^j + T[S'] \cdot t^k$. Since S' is a finite T'-module and T is a finite T'-module, $T[S']$ is a finite T-module. It follows that S is contained in a finite T-module, so the result holds.

(iii) implies (i). Because an ideal of the principal class is generated by analytically independent elements, this follows from (ii) implies (i).

5. Concluding Remarks

1. Ideally one would like to have a closure operation "$\tilde{\ }$" on ideals, similar to integral closure, so that the sets $\operatorname{Ass} R/\widetilde{J^n}$ would take the value $\tilde{A}^*(I)$ for all large n. This would allow for a theory of asmptotic R-sequences and unmixed local rings that would be entirely analogous to that of asymptotic (prime) sequences and quasi-unmixed local rings developed in [5, 7, 19]. In the local case

$$\bar{J} = \bigcap \varphi_{Q^*}^{-1}\overline{(JR^* + Q^*/Q^*)}, \qquad Q^* \text{ minimal}$$

where $\varphi_{Q^*}: R \to R^*/Q^*$ is the natural map. This suggests trying

$$\tilde{J} = \bigcap \varphi_{Q^*}^{-1}\overline{(JR^* + Q^*/Q^*)}, \qquad Q^* \in \operatorname{Ass} R^*;$$

however, at this time, there does not appear to be any natural way of expressing this ideal in terms of properties intrinsic to J and R.

2. It should be mentioned that some of our results are similar to those obtained independently by the authors in [8]. In [8], McAdam and Ratliff introduce the set $A_*(I) = \{P \supseteq I \mid (R_P)^*$ has a prime divisor of zero Q^*, with $P(R_P)^*$ minimal over $I(R_P)^* + Q^*\}$. Using this set of primes they define the notion of essential sequence. Now, in general $A_*(I) \subseteq \tilde{A}^*(I)$ and the containment may be proper. Moreover, $\bar{A}^*(I)$ need not be contained in $A_*(I)$. If, however, I is an ideal generated by an asymptotic R-sequence, then one can show $\bar{A}^*(I) \subseteq A_*(I) = \tilde{A}^*(I) \subseteq A^*(I)$. It follows that asymptotic R-sequences are essential sequences and conversely. $A_*(I)$ appears to be an important set of primes and the interested reader is urged to consult [8] for further details.

3. Condition (iii) in Theorem 4.1 cannot be weakened to height $I < \dim R$. This can be seen as follows: Let R be an unmixed local ring and $I \subseteq R$ an ideal with height $I < \dim R$ and analytic spread $I = \dim R$. By ([7, Proposition 4.1]) $M \in \bar{A}^*(I) \subseteq \tilde{A}^*(I)$. Suppose S were a finite T-module. Then:

(i) There exists $k > 0$ such that $(I^k : \langle M \rangle)^n = I^{kn} : \langle M \rangle$ for all $n \geqslant 1$ (as in the proof of Theorem 2.5).

(ii) $S \subseteq \bar{T} =$ integral closure of $T = \sum_n \overline{I^n} t^n$.

The second condition implies $I^n \subseteq I^n : \langle M \rangle \subseteq \overline{I^n}$ for all $n \geqslant 1$. In particular, for k as in (i), $I^k \subseteq I^k : \langle M \rangle \subseteq \overline{I^k}$, so $M \in A^*(I^k : \langle M \rangle)$. That is, for all large n, M consists of zero divisors modulo $(I^k : \langle M \rangle)^n = I^{kn} : \langle M \rangle$, a contradiction.

4. Schenzel's problem has an integral closure analogue. The same circle of ideas used in the proof of Theorem 4.1 may be applied to prove the following:

THEOREM. *Let (R, M) be a local ring. The following are equivalent*:

(i) *R is quasi-unmixed.*

(ii) *For each ideal of the principal class I, with* height $I < \dim R$, $S \subseteq \bar{T}$.

(iii) *For each ideal I with analytic spread* $I < \dim R$, $S \subseteq \bar{T}$.

Only the following modifications in the proof of Theorem 4.1 are required:

(a) One works with minimal primes in the completion, rather than all prime divisors.

(b) One uses $\bar{A}^*(I)$ rather than $\tilde{A}^*(I)$.

(c) One invokes the integral closure analogue of Nishimura's result: Namely, that the transform of an ideal I is an integral extension if and only if for all $P \supseteq I$, $(R_P)^*$ does not have a depth one minimal prime divisor of zero.

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